EXAMPLE 4S.2

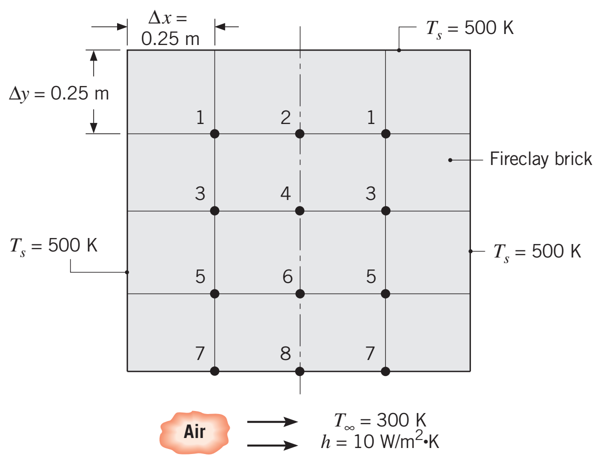
A large industrial furnace is supported on a long column of fireclay brick, which is 1 m × 1 m on a side. During steady-state operation, installation is such that three surfaces of the column are maintained at 500 K while the remaining surface is exposed to an airstream for which $T_\infty = 300$ K and $h = 10$ W/m$^2\cdot$K. Using a grid of $\Delta x = \Delta y = 0.25$ m, determine the two-dimensional temperature distribution in the column and the heat rate to the airstream per unit length of column.

SOLUTION

Known: Dimensions and surface conditions of a support column.

Find: Temperature distribution and heat rate per unit length.

Schematic:



Assumptions:

1. Steady-state conditions.
2. Two-dimensional conduction.
3. Constant properties.
4. No internal heat generation.

Properties: Table A.3, fireclay brick ($T \approx 478$ K): $k = 1$ W/m $\cdot$ K.

Analysis: The prescribed grid consists of 12 nodal points at which the temperature is unknown. However, the number of unknowns is reduced to 8 through symmetry, in which case the temperature of nodal points to the left of the symmetry line must equal the temperature of those to the right.

Nodes 1, 3, and 5 are interior points for which the finite-difference equations may be inferred from Equation 4.29. Hence

Node 1: $T_2 + T_3 + 1000 - 4T_1 = 0$
Node 3: $T_1 + T_4 + T_5 + 500 - 4T_3 = 0$
Node 5: $T_3 + T_6 + T_7 + 500 - 4T_5 = 0$

Equations for points 2, 4, and 6 may be obtained in a like manner or, since they lie on a symmetry adiabat, by using Equation 4.42 with $h = 0$. Hence

Node 2: $2T_1 + T_4 + 500 - 4T_2 = 0$
Node 4: $T_2 + 2T_3 + T_6 - 4T_4 = 0$
Node 6: $T_4 + 2T_5 + T_8 - 4T_6 = 0$

From Equation 4.42 and the fact that $h\,\Delta x/k = 2.5$, it also follows that

Node 7: $2T_5 + T_8 + 2000 - 9T_7 = 0$
Node 8: $2T_6 + 2T_7 + 1500 - 9T_8 = 0$

Having the required finite-difference equations, the temperature distribution will be determined by using the Gauss–Seidel iteration method. Referring to the arrangement of finite-difference equations, it is evident that the order is already characterized by diagonal dominance. This behavior is typical of finite-difference solutions to conduction problems. We therefore begin with step 2 and express the equations in explicit form

$$
\begin{aligned}
T_1^{(k)} &= 0.25T_2^{(k-1)} + 0.25T_3^{(k-1)} + 250 \\
T_2^{(k)} &= 0.50T_1^{(k)} + 0.25T_4^{(k-1)} + 125 \\
T_3^{(k)} &= 0.25T_1^{(k)} + 0.25T_4^{(k-1)} + 0.25T_5^{(k-1)} + 125 \\
T_4^{(k)} &= 0.25T_2^{(k)} + 0.50T_3^{(k)} + 0.25T_6^{(k-1)} \\
T_5^{(k)} &= 0.25T_3^{(k)} + 0.25T_6^{(k-1)} + 0.25T_7^{(k-1)} + 125 \\
T_6^{(k)} &= 0.25T_4^{(k)} + 0.50T_5^{(k)} + 0.25T_8^{(k-1)} \\
T_7^{(k)} &= 0.2222T_5^{(k)} + 0.1111T_8^{(k-1)} + 222.22 \\
T_8^{(k)} &= 0.2222T_6^{(k)} + 0.2222T_7^{(k)} + 166.67
\end{aligned}
$$

Having the finite-difference equations in the required form, the iteration procedure may be implemented by using a table that has one column for the iteration (step) number and a column for each of the nodes labeled as T_i. The calculations proceed as follows:

1. For each node, the initial temperature estimate is entered on the row for $k = 0$. Values are selected rationally to reduce the number of required iterations.
2. Using the N finite-difference equations and values of T_i from the first and second rows, the new values of T_i are calculated for the first iteration ($k = 1$). These new values are entered on the second row.

3. This procedure is repeated to calculate $T_i^{(k)}$ from the previous values of $T_i^{(k-1)}$ and the current values of $T_i^{(k)}$, until the temperature difference between iterations meets the prescribed criterion, $\varepsilon \leq 0.2$ K, at every nodal point.

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	480	470	440	430	400	390	370	350
1	477.5	471.3	451.9	441.3	428.0	411.8	356.2	337.3
2	480.8	475.7	462.5	453.1	432.6	413.9	355.8	337.7
3	484.6	480.6	467.6	457.4	434.3	415.9	356.2	338.3
4	487.0	482.9	469.7	459.6	435.5	417.2	356.6	338.6
5	488.1	484.0	470.8	460.7	436.1	417.9	356.7	338.8
6	488.7	484.5	471.4	461.3	436.5	418.3	356.9	338.9
7	489.0	484.8	471.7	461.6	436.7	418.5	356.9	339.0
8	489.1	485.0	471.9	461.8	436.8	418.6	356.9	339.0

The results given in row 8 are in excellent agreement with those that would be obtained by an exact solution of the matrix equation, although better agreement could be obtained by reducing the value of ε. However, given the approximate nature of the finite-difference equations, the results still represent approximations to the actual temperatures. The accuracy of the approximation may be improved by using a finer grid (increasing the number of nodes).

The heat rate from the column to the airstream may be computed from the expression

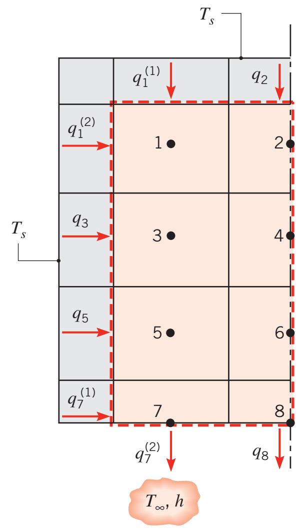
$$\left(\frac{q}{L}\right) = 2h\left[\left(\frac{\Delta x}{2}\right)(T_s - T_\infty) + \Delta x\,(T_7 - T_\infty) + \left(\frac{\Delta x}{2}\right)(T_8 - T_\infty)\right]$$

where the factor of 2 outside the brackets originates from the symmetry condition. Hence

$$\left(\frac{q}{L}\right) = 2 \times 10\ \mathrm{W/m^2 \cdot K}\ [0.125\ \mathrm{m}\ (200\ \mathrm{K})$$
$$+\ 0.25\ \mathrm{m}\ (56.9\ \mathrm{K}) + 0.125\ \mathrm{m}\ (39.0\ \mathrm{K})] = 882\ \mathrm{W/m} \quad \triangleleft$$

Comments:

1. To ensure that no errors have been made in formulating the finite-difference equations or in effecting their solution, a check should be made to verify that the results satisfy conservation of energy for the nodal network. For steady-state conditions, the requirement dictates that the rate of energy inflow be balanced by the rate of outflow *for a control surface surrounding all nodal regions whose temperatures have been evaluated.*



For the one-half symmetrical section shown schematically above, it follows that conduction into the nodal regions must be balanced by convection from the regions. Hence

$$q_1^{(1)} + q_1^{(2)} + q_2 + q_3 + q_5 + q_7^{(1)} = q_7^{(2)} + q_8$$

The cumulative conduction rate is then

$$\frac{q_{\text{cond}}}{L} = k\left[\Delta x \frac{(T_s - T_1)}{\Delta y} + \Delta y \frac{(T_s - T_1)}{\Delta x} + \frac{\Delta x}{2}\frac{(T_s - T_2)}{\Delta y}\right.$$

$$\left. + \Delta y \frac{(T_s - T_3)}{\Delta x} + \Delta y \frac{(T_s - T_5)}{\Delta x} + \frac{\Delta y}{2}\frac{(T_s - T_7)}{\Delta x}\right]$$

$$= 192.1 \text{ W/m}$$

and the convection rate is

$$\frac{q_{\text{conv}}}{L} = h\left[\Delta x (T_7 - T_\infty) + \frac{\Delta x}{2}(T_8 - T_\infty)\right] = 191.0 \text{ W/m}$$

Agreement between the conduction and convection rates is excellent, confirming that mistakes have not been made in formulating and solving the finite-difference equations. Note that convection transfer from the entire bottom surface (882 W/m) is obtained by adding transfer from the edge node at 500 K (250 W/m) to that from the interior nodes (191.0 W/m) and multiplying by 2 from symmetry.

2. Although the computed temperatures satisfy the finite-difference equations, they do not provide us with the exact temperature field. Remember that the equations are approximations whose accuracy may be improved by reducing the grid size (increasing the number of nodal points).